Thermodynamics of the ideal Fermi gas

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Note: Usually, we work in natural units where $\hbar = c = k_B = 1$. However, in this document I explicitly include all of these so that you can see where they usually pop up. For your codes, please continue to use natural units. This is equivalent to simply omitting any $\hbar$'s, c's, and k_B's that you see here.

1. WHY IS THIS IMPORTANT?

Nucleons (protons and neutrons) are particles with spin-$\frac{1}{2}$, which means that they are fermions. Consequently, each distinct state in a system can be occupied by only one distinct nucleon, which significantly affects the way these systems are described thermodynamically. Particles with Fermi statistics can be described within the framework of an ideal Fermi gas. Even though nucleons interact and therefore are not an *ideal* gas, the concept of an ideal Fermi gas is a necessary starting point when describing nuclear matter. In the case of nuclear matter as found in nuclei or in neutron stars, one can assume that the temperature of the system is zero, which considerably simplifies the expressions and which is why we will discuss the ideal Fermi gas at zero temperature. On the other hand, in heavy-ion collisions high temperatures can be reached, and the description in this case must involve finite-temperature contributions to the thermodynamic variables (which we will also discuss).

Note that the temperature of a neutron star $T_{\rm NS}$ is on the order of 10^6 K, which ordinarily is by no means small. For example, the core temperature of the Sun is about 1.5×10^7 K, while its surface temperature is about 5.7×10^3 K. To evaluate what "small" means in a given context, one needs to compare a given quantity to other quantities present in a given system. One can calculate that the energy associated with the temperature of a neutron star is approximately equal $k_B T_{\rm NS} \approx 10^{-4}$ MeV (where we used $k_B = 8.6173303 \times 10^{-11}$ MeV/K). Compared to the energy associated with

the sole fact that a nucleon has a mass of $m_N \approx 939$ MeV, this really is negligible and for all intents and purposes neutron stars are at $T = 0$ from the perspective of nuclear matter. (An even better example would be to compare with typical momenta of nucleons in a neutron star, as effects due to temperature can be understood in terms of changes to how momentum states are occupied, but we will only learn how to calculate the momenta in what follows.) There are some important exceptions from this. For example, neutron stars that are in the process of being formed after a core-collapse of a supernova (known as protoneutron stars) are characterized by temperatures on the order of 10 MeV, and the nuclear matter in neutron star mergers is speculated to reach up to about 50–100 MeV. In heavy-ion collisions, depending on the beam energy, temperatures from several to hundreds of MeV can be achieved.

2. FREE QUANTUM GAS IN A BOX

We want to consider a free quantum gas of a given species of particles.. Let us assume that this gas occupies some volume V. Specifically, let us consider the Schrödinger equation for a free particle in a box with a side length L (where $L^3 = V$),

$$\hat{H}\psi(x,y,z) = -\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) = E\psi(x,y,z) \ , \tag{1}$$

with the following boundary conditions for the wave function $\psi(x,y,z)$:

$$\psi\big(x=0,y,z\big) = \psi\big(x=L,y,z\big) = \psi\big(x,y=0,z\big) = \psi\big(x,y=L,z\big) = \psi\big(x,y,z=0\big) = \psi\big(x,y,z=L\big) = 0 \ . \tag{2}$$

Because there are no interactions or any other factors that would correlate the behavior of the wave function in any two of the three space directions, the wave function should factorize into

$$\psi(x,y,z) = \psi_x(x)\psi_y(y)\psi_z(z) \ . \tag{3}$$

Furthermore, we know from quantum mechanics that a free quantum particle should be described by a plane wave, i.e.,

$$\psi_\alpha(\alpha) = A_\alpha \sin k_\alpha\alpha + B_\alpha \cos k_\alpha\alpha \ , \qquad \alpha = x,y,z \ . \tag{4}$$

The boundary conditions at $\alpha = 0$, Eq. (2), immediately give $B_\alpha = 0$ for all values of α. The other set of boundary conditions yields a constraint on the values of the wavenumbers k_α,

$$A_\alpha \sin k_\alpha L = 0 \ . \tag{5}$$

This is only satisfied if the argument of the sine function is an integer multiple of π, i.e.,

$$k_\alpha L = n_\alpha \pi \qquad \Rightarrow \qquad k_\alpha = \frac{n_\alpha \pi}{L} \ . \tag{6}$$

The wave function is then given by

$$\psi_{n_x,n_y,n_z}(x,y,z) = A \sin\left(\frac{n_x\pi}{L}x\right)\sin\left(\frac{n_y\pi}{L}y\right)\sin\left(\frac{n_z\pi}{L}z\right) \ , \tag{7}$$

and the eigenenergies (that is the eigenvalues corresponding to this solution) are given by

$$E_{n_x,n_y,n_z} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2+n_y^2+n_z^2}{L^2}\right) \ . \tag{8}$$

3. OCCUPATION IN THE MOMENTUM SPACE

As shown above, the wavenumbers of the component wavefunctions ψ_x, ψ_y, ψ_z must satisfy (6). We want to consider a phase space, that is a $6N$-dimensional space of all possible positions and momenta of N particles. In terms of momenta p_α, the conditions in Eq. (6) become

$$p_x = \pm\frac{\pi\hbar}{L} \ , \pm\frac{2\pi\hbar}{L} \ , \dots \ , \tag{9}$$

and similarly for p_y and p_z (this follows from the fact that the momentum is related to the de Broglie wavelength of a particle through $\lambda_\alpha = 2\pi\hbar/p_\alpha$, and the wavelength is related to the wave number by $k_\alpha = 2\pi/\lambda_\alpha$).

In the following we want to concentrate on fermions, i.e., electrons, protons, neutrons, etc. In this case, any given particle not only has to assume one of the discrete (quantized) momentum values from Eq. (9), but also no two particles can occupy the same state. From Eq. (7) we can see that the $+n_\alpha\pi\hbar/L$ and $-n_\alpha\pi\hbar/L$ momentum states are equivalent (i.e., they lead to an exact same wave function, save for the overall sign which is irrelevant), so that only one of these can be occupied at a time. If one considers this in terms of a region in the phase space, one can define a "unit volume" taken up in the momentum space by one fermion as follows,

$$V_0 = \frac{(2\pi\hbar)^3}{L^3} \ . \tag{10}$$

I.e., this is the volume of the phase space "blocked out" by one fermion (note that V_0 doesn't specify *where* that blocked out volume is – all we obtain here is an idea of how much of the phase space is available/unavailable).

Let us now consider some particular magnitude p of a momentum vector $\boldsymbol{p}$. The number of available states in the volume of the momentum space spanned by all vectors such that $|\boldsymbol{p}| = p$ is given by

$$N_p = \frac{\frac{4}{3}\pi p^3}{V_0} = \frac{\frac{4}{3}\pi p^3}{(2\pi\hbar)^3} L^3 \ . \tag{11}$$

Since each such state can be occupied by states with different values of spin, the total number of states is

$$N = g\frac{\frac{4}{3}\pi p^3}{(2\pi\hbar)^3} L^3 \ , \tag{12}$$

where $g = 2s + 1$ is the spin degeneracy, that is the number of available spin states per a given momentum for a particle with spin s (for example, for an electron or a proton we have $g = 2$). Then the number of states per unit volume is given by

$$n = g\frac{\frac{4}{3}\pi p^3}{(2\pi\hbar)^3} \ . \tag{13}$$

From this, we can obtain the number of states per a given momentum interval by differentiating,

$$\frac{dn}{dp} = g\frac{4\pi p^2}{(2\pi\hbar)^3} \ , \tag{14}$$

which leads to

$$dn = g\frac{4\pi p^2}{(2\pi\hbar)^3}\ dp \ . \tag{15}$$

4. THERMODYNAMIC OBSERVABLES AT $T = 0$

So far, we have shown that for free fermions the number of states per unit volume dn available at momentum p is given by Eq. (15). At zero temperature, the considered system must be in the ground state. This means that the system occupies the combination of states with the lowest possible energy, which for a free Fermi gas in a box means that the particles fill the energy states from the lowest available state ($p \approx 0$) up, until all particles are assigned a state. Consequently, there will be a state with the highest magnitude of momentum in a given system, which is called the Fermi momentum p_F. All states with momenta from 0 to p_F are occupied.

4.1. Number density

We can then integrate the density of states in the momentum space, Eq. (15), over all occupied momenta to obtain the number density (i.e., the number of particles per unit volume),

$$n = g\int_0^{p_F} dp\ \frac{p^2}{2\pi^2\hbar^3} = \frac{gp_F^3}{6\pi^2\hbar^3} \ . \tag{16}$$

Note that one can solve Eq. (16) for p_F in terms of the number density n,

$$p_F = \left(\frac{6\pi^2\hbar^3 n}{g}\right)^{1/3} . \tag{17}$$

One can see that the higher the number density n, the higher is the value of the maximum momentum occupied in the system. This makes sense: if we have a number of particles in a given region of space, occupying all momenta until some momentum p_F, and then we add yet another particle into the same region (so that now we have $n' > n$), that particle must take the next available unoccupied momentum state and as a result we have a new, larger value of the Fermi momentum $p'_F > p_F$.

4.2. Energy density

With Eq. (15), we can calculate not only the number density, but also the energy density by summing over all possible particle energies ε multiplied by the number of particles at that energy. For a relativistic free particle, we have $\varepsilon = \sqrt{m^2c^4 + p^2c^2}$. Therefore, the energy density is given by

$$\mathcal{E}\big|_{T=0} = \int dn\ \varepsilon = \int dn\ \sqrt{m^2c^4 + p^2c^2} = \frac{g}{2\pi^2\hbar^3}\int_0^{p_F} dp\ p^2\ \sqrt{m^2c^4 + p^2c^2} \tag{18}$$

$$= \frac{g}{2\pi^2\hbar^3}\int_0^{p_F} dp\ p^2\ mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} . \tag{19}$$

Substituting

$$u = \frac{p}{mc} \qquad \Rightarrow \qquad dp = mc\ du\ , \tag{20}$$

we obtain

$$\mathcal{E}\big|_{T=0} = \frac{gm^4c^5}{2\pi^2\hbar^3}\int_0^{\frac{p_E}{mc}} du\ u^2\ \sqrt{1+u^2} \tag{21}$$

$$= \frac{gm^4c^5}{16\pi^2\hbar^3}\left[\sqrt{1+\frac{p_F^2}{m^2c^2}}\left(\frac{p_F}{mc} + \frac{2p_F^3}{m^3c^3}\right) - \sinh^{-1}\left(\frac{p_F}{mc}\right)\right] \tag{22}$$

$$= \frac{g}{16\pi^2\hbar^3}\left[\sqrt{m^2c^4 + p_F^2c^2}\left(p_F m^2c^2 + 2p_F^3\right) - m^4c^5\sinh^{-1}\left(\frac{p_F c}{mc^2}\right)\right] . \tag{23}$$

Since the Fermi energy, that is the highest energy in the system, is given by $\varepsilon_F = \sqrt{m^2c^4 + p_F^2c^2}$, we can also write

$$\mathcal{E}\big|_{T=0} = \frac{g}{16\pi^2\hbar^3}\left[\varepsilon_F p_F\left(m^2c^2 + 2p_F^2\right) - m^4c^5\sinh^{-1}\left(\frac{p_F c}{mc^2}\right)\right] \tag{24}$$

$$= \frac{g}{16\pi^2\hbar^3}\left[\varepsilon_F p_F\left(m^2c^2 + p_F^2 + p_F^2\right) - m^4c^5\sinh^{-1}\left(\frac{p_F c}{mc^2}\right)\right] \tag{25}$$

$$= \frac{g}{16\pi^2\hbar^3}\left[\varepsilon_F p_F\left(\frac{\varepsilon_F^2}{c^2} + p_F^2\right) - m^4c^5\sinh^{-1}\left(\frac{p_F c}{mc^2}\right)\right] \tag{26}$$

$$= \frac{g}{16\pi^2\hbar^3}\left[\frac{\varepsilon_F^3 p_F}{c^2} + \varepsilon_F p_F^3 - m^4c^5\sinh^{-1}\left(\frac{p_F c}{mc^2}\right)\right] . \tag{27}$$

Because $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$, we can further rewrite the above as

$$\mathcal{E}\big|_{T=0} = \frac{g}{16\pi^2\hbar^3}\left[\frac{\varepsilon_F^3 p_F}{c^2} + \varepsilon_F p_F^3 - m^4c^5\ln\left(\frac{\varepsilon_F + p_F c}{mc^2}\right)\right] . \tag{28}$$

Finally, note that we can also write $p_F^2 = \varepsilon_F^2/c^2 - m^2c^2$, by means of which the energy density can be also written as

$$\mathcal{E}\big|_{T=0} = \frac{g}{16\pi^2\hbar^3}\left[\frac{2\varepsilon_F^3 p_F}{c^2} - \varepsilon_F p_F m^2c^2 - m^4c^5\ln\left(\frac{\varepsilon_F + p_F c}{mc^2}\right)\right] . \tag{29}$$

All of these forms are obviously equivalent, and you can see different forms used depending on the personal preference of a given author.

4.3. Pressure

To get the pressure, let us recall the first law of thermodynamics,

$$dE = TdS - PdV + \mu dN \ , \tag{30}$$

where E is the total energy of the system, T is the temperature, S is the entropy, P is pressure, V is volume, μ is the chemical potential, and N is the number of particles. Let us assume that the number of particles is constant, $dN = 0$. Then we can divide both sides of the above equation by N,

$$d\left(\frac{E}{N}\right) = T\ d\left(\frac{S}{N}\right) - P\ \frac{dV}{N} \ . \tag{31}$$

On the other hand, the number density is given by

$$n = \frac{N}{V} \qquad \Rightarrow \qquad V = \frac{N}{n} \ . \tag{32}$$

Since $N = \text{const}$, this leads to

$$dV = -\frac{N}{n^2} dn \ , \tag{33}$$

so that now we have

$$d\left(\frac{E}{N}\right) = T\ d\left(\frac{S}{N}\right) + \frac{P}{n^2}\ dn \ . \tag{34}$$

Since $T = 0$, the first term on the right hand side vanishes and we obtain

$$P\big|_{T=0} = n^2 \frac{d\left(\frac{E}{N}\right)}{dn} \ . \tag{35}$$

Finally, one can always write $\frac{E}{N} = \frac{E}{V}\frac{V}{N} = \frac{\mathcal{E}}{n}$, so that in the end

$$P\big|_{T=0} = n^2 \frac{d\left(\frac{\mathcal{E}}{n}\right)}{dn} \ . \tag{36}$$

Using Eq. (36) by brute force is not advised (although it should get you the correct result if you keep track of all terms). Instead, let us first notice that we can rewrite

$$P\big|_{T=0} = n\frac{d\mathcal{E}}{dn} - \mathcal{E} \ . \tag{37}$$

Furthermore,

$$\frac{d\mathcal{E}}{dn} = \frac{dp_F}{dn}\frac{d\mathcal{E}}{dp_F} = \frac{p_F}{3n}\frac{d\mathcal{E}}{dp_F} \ , \tag{38}$$

where we have used Eq. (17). Let us now use the fact that $\mathcal{E}$ is from the definition given by Eq. (18) and that for any function the following is true:

$$\frac{d}{dx_0}\int_0^{x_0} dx\ f(x) = f(x_0) \ . \tag{39}$$

Consequently,

$$\frac{d\mathcal{E}}{dp_F} = \frac{g}{2\pi^2\hbar^3} p_F^2 \varepsilon_F \ , \tag{40}$$

and so altogether

$$\frac{d\mathcal{E}}{dn} = \frac{1}{n}\frac{g}{6\pi^2\hbar^3} p_F^3 \varepsilon_F = \varepsilon_F \ , \tag{41}$$

where we again used Eq. (17). $d\mathcal{E}/dn \equiv \mu$, so that we also obtain that for the ideal Fermi gas at zero temperature the Fermi energy is equal to the chemical potential, $\varepsilon_F = \mu$.

Altogether, we arrive at

$$P\big|_{T=0} = n\varepsilon_F - \mathcal{E} = \frac{g}{6\pi^2\hbar^3} p_F^3 \varepsilon_F - \mathcal{E} \ . \tag{42}$$

Together with Eq. (28), this yields

$$P\big|_{T=0} = \frac{g}{6\pi^2\hbar^3} p_F^3 \varepsilon_F - \frac{g}{16\pi^2\hbar^3}\left[\frac{\varepsilon_F^3 p_F}{c^2} + \varepsilon_F p_F^3 - m^4c^5 \ln\left(\frac{\varepsilon_F + p_F c}{mc^2}\right)\right] \tag{43}$$

$$= \frac{g}{16\pi^2\hbar^3}\left[\frac{5}{3} p_F^3 \varepsilon_F - \frac{\varepsilon_F^3 p_F}{c^2} + m^4c^5 \ln\left(\frac{\varepsilon_F + p_F c}{mc^2}\right)\right] \tag{44}$$

$$= \frac{g}{16\pi^2\hbar^3}\left[\frac{5}{3} p_F^3 \varepsilon_F - \frac{\left(m^2c^4 + p_F^2c^2\right)\varepsilon_F p_F}{c^2} + m^4c^5 \ln\left(\frac{\varepsilon_F + p_F c}{mc^2}\right)\right] \tag{45}$$

$$= \frac{g}{16\pi^2\hbar^3}\left[\frac{2}{3}\varepsilon_F p_F^3 - \varepsilon_F p_F m^2c^2 + m^4c^5 \ln\left(\frac{\varepsilon_F + p_F c}{mc^2}\right)\right] \ . \tag{46}$$

Note that from the first law of thermodynamics, we have by definition

Finally, let us note that the pressure of a system with an isotropic distribution of momenta (e.g., in an ideal gas where there is no reason for the distribution not to be isotropic) can be also written in the form which comes from averaging over the three spatial components of the energy-momentum tensor,

$$P = \int dn \ \frac{pv}{3} \ . \tag{47}$$

At $T = 0$, this becomes

$$P\big|_{T=0} = \frac{g}{2\pi^2\hbar^3} \int_0^{p_F} dp \ p^2 \ \frac{p^2c^2}{3\varepsilon} \ , \tag{48}$$

where we have used the fact that the velocity is given by $v = pc^2/\varepsilon$. For details, see Appendix A.

4.3.1. Non-relativistic limit

In the non-relativistic limit, where $p_F c \ll mc^2$, we can write

$$\mathcal{E}\big|_{T=0} = \frac{g}{2\pi^2\hbar^3} \int_0^{p_F} dp \ p^2 \ \sqrt{m^2c^4 + p^2c^2} \approx \frac{g}{2\pi^2\hbar^3} \int_0^{p_F} dp \ p^2 \ mc^2 = \frac{gmc^2 p_F^3}{6\pi^2\hbar^3} \ . \tag{49}$$

Similarly, using Eq. (48), we can obtain for the pressure

$$P\big|_{T=0} = \frac{g}{2\pi^2\hbar^3} \int_0^{p_F} dp \ p^2 \ \frac{p^2c^2}{3\sqrt{m^2c^4 + p^2c^2}} \approx \frac{g}{6\pi^2\hbar^3} \int_0^{p_F} dp \ \frac{p^4}{m} = \frac{g p_F^5}{30 m\pi^2\hbar^3} \ . \tag{50}$$

With Eq. (17), we can rewrite both of the above equations in terms of the mass density $\rho = nm$,

$$\mathcal{E}\big|_{T=0} = \rho c^2 \quad \text{and} \quad P\big|_{T=0} = \frac{1}{5m^{8/3}} \left(\frac{6\pi^2\hbar^3}{g}\right)^{2/3} \rho^{5/3} \ . \tag{51}$$

Defining

$$K^{(\text{nonrel})} = \frac{1}{5m^{8/3}} \left(\frac{6\pi^2\hbar^3}{g}\right)^{2/3} \ , \tag{52}$$

we can write

$$P\big|_{T=0} = K^{(\text{nonrel})} \rho^{5/3} \ . \tag{53}$$

4.3.2. Ultra-relativistic limit

In the ultra-relativistic limit, where $p_F c \gg mc^2$, we have

$$\mathcal{E}\big|_{T=0} = \frac{g}{2\pi^2\hbar^3}\int_0^{p_F} dp\ p^2\ \sqrt{m^2c^4+p^2c^2} \approx \frac{g}{2\pi^2\hbar^3}\int_0^{p_F} dp\ p^2\ pc = \frac{gcp_F^4}{8\pi^2\hbar^3} = \frac{gc}{8\pi^2\hbar^3}\left(\frac{6\pi^2\hbar^3 n}{g}\right)^{4/3} \tag{54}$$

$$= \frac{gc}{8\pi^2\hbar^3}\left(\frac{6\pi^2\hbar^3}{gm}\right)^{4/3}\rho^{4/3} \tag{55}$$

and

$$P\big|_{T=0} = \frac{g}{2\pi^2\hbar^3}\int_0^{p_F} dp\ p^2\ \frac{p^2c^2}{3\sqrt{m^2c^4+p^2c^2}} \approx \frac{g}{6\pi^2\hbar^3}\int_0^{p_F} dp\ p^2\ pc = \frac{gcp_F^4}{24\pi^2\hbar^3} = \frac{gc}{24\pi^2\hbar^3}\left(\frac{6\pi^2\hbar^3 n}{g}\right)^{4/3} \tag{56}$$

$$= \frac{gc}{24\pi^2\hbar^3}\left(\frac{6\pi^2\hbar^3}{gm}\right)^{4/3}\rho^{4/3}\ . \tag{57}$$

Note that in this case $P = \mathcal{E}/3$. We can again define

$$K^{(\mathrm{rel})} = \frac{gc}{24\pi^2\hbar^3}\left(\frac{6\pi^2\hbar^3}{gm}\right)^{4/3}\ , \tag{58}$$

so that

$$P\big|_{T=0} = K^{(\mathrm{rel})}\rho^{4/3}\ . \tag{59}$$

4.3.3. Polytropic equation of state

The results from the two previous subsections suggest that, in general, one can write the pressure of an ideal gas at $T = 0$ in the form

$$P\big|_{T=0} = K\rho^{\Gamma}\ , \tag{60}$$

where K and Γ depend on the underlying physics. An equation of state of this form is called a polytropic equation of state, K is known as the polytropic constant,. and Γ is referred to as the polytropic exponent or adiabatic index.

5. THERMODYNAMIC OBSERVABLES AT FINITE T

Having calculated the above quantities for an ideal Fermi gas at $T = 0$, we can now generalize our discussion. We already saw that a complete description of a system can be derived as long as the number density in the phase space of a species of particles,

$$dn \equiv \frac{d\mathcal{N}}{d^3x d^3p}\ , \tag{61}$$

is known. An equivalent way of describing the system is to specify a dimensionless distribution function in the phase space, $f(x, p, t)$, defined as

$$\frac{d\mathcal{N}}{d^3x d^3p} = \frac{g}{(2\pi\hbar)^3}f\ , \tag{62}$$

where g is the degeneracy factor and $(2\pi\hbar)^3$ is the volume of a cell in the momentum space (as we've shown). The function f gives an average occupation number in a cell of the phase space.

The number density of a species of particles is then obtained by integrating over all momenta,

$$n = \int d^3p\ \frac{d\mathcal{N}}{d^3x d^3p} = g\int \frac{d^3p}{(2\pi\hbar)^3}\ f\ . \tag{63}$$

Similarly, the energy density is given by

$$\mathcal{E} = \int d^3p\ \varepsilon \frac{d\mathcal{N}}{d^3x d^3p} = g \int \frac{d^3p}{(2\pi\hbar)^3}\ \varepsilon\ f\ , \tag{64}$$

and the pressure of the system (with an isotropic distribution of momenta) is given by

$$P = \int d^3p\ \frac{pv}{3} \frac{d\mathcal{N}}{d^3x d^3p} = \frac{g}{3} \int \frac{d^3p}{(2\pi\hbar)^3}\ pv\ f\ , \tag{65}$$

where again the velocity is given by $v = \frac{pc^2}{\varepsilon}$.

For spherically symmetrical distributions, the above formulas can be further rewritten as

$$n = \frac{g}{2\pi^2\hbar^3} \int_0^\infty dp\ p^2\ f\ , \tag{66}$$

$$\mathcal{E} = \frac{g}{2\pi^2\hbar^3} \int_0^\infty dp\ p^2\ \varepsilon\ f\ , \tag{67}$$

and

$$P = \frac{gc^2}{6\pi^2\hbar^3} \int_0^\infty dp\ \frac{p^4}{\varepsilon}\ f\ . \tag{68}$$

The above are very concise and specific expressions for the density, energy density, and pressure of the system. The apparent simplicity is due to hiding all the relevant information in f: actually using these formulas requires one to know what the functional form of f is.

5.1. The Boltzmann factor

We want to know what form f takes in the case of particles with quantum properties. There are two cases here: fermions (which cannot share the same quantum state) and bosons (which can share the same quantum state). The starting point of the derivation is the statement that the Boltzmann factor $e^{-\beta(\varepsilon-\mu)}$, where $\beta = (k_B T)^{-1}$, is a *statistical weight* for a system to occupy an energy state ε given temperature T and chemical potential μ.

This fact is usually covered in depth in a statistical mechanics or thermal physics class, but it can also be understood on a somewhat intuitive level. First, let's note that β is always positive. Then, the behavior of the whole factor depends strongly on the sign of the term in the bracket, $(\varepsilon - \mu)$. If $\varepsilon > \mu$, then $\beta(\varepsilon - \mu) > 0$ and the overall exponent is negative, leading to *small* values of $e^{-\beta(\varepsilon-\mu)}$. This means that states with energies higher than the chemical potential are *suppressed*. Note that the bigger the difference $(\varepsilon - \mu)$ is, the *more suppressed* is the corresponding state, and *vice versa*. Note also that for a given value of $(\varepsilon - \mu)$, the state is *more suppressed* for smaller values of temperature T, and *less suppressed* for larger values of T. Altogether, what this means is that states with energies higher than the chemical potential of the system are suppressed, although less so if the temperature is high: at higher temperatures, the probability that *some* of the particles will be "kicked" into a high energy state due to the random thermal motion of the system is simply higher. A similar reasoning can be made for energy states satisfying $\varepsilon < \mu$: in that case, the overall exponent in the Boltzmann factor is positive, leading to *large* values of $e^{-\beta(\varepsilon-\mu)}$. This means that states with values of energy smaller than μ are statistically *enhanced*.

All of this hangs together in a somewhat inter-dependent way. For example, the chemical potential is generally a function of temperature. If a system is characterized by a small chemical potential μ, this means that it doesn't take much energy to add another particle to the system. This is either because the energies occupied in the system are generally low, or because the temperature is high which makes it easier to add highly energetic particles. Conversely, if the chemical potential is high, it takes a lot of energy to add another particle to the system. That's the case when the target energy states are high. Now again the temperature also plays a role here: in general, for a given set of target states, if the temperature is high, then it's easier to populate high energy states and consequently the chemical potential of the system is lower. If the temperature is low, the opposite is the case. The Boltzmann factor contains all of these interdependent possibilities in it.

If we accept the fact that the Boltzmann factor is a statistical weight (i.e., it tells us how likely it is that a particle occupies a state of energy ε), then we can use it to calculate statistical averages. Let us consider the probability of

occupying a specific single-particle energy level ε, and let us consider that we want N particles to occupy that energy level. The Boltzmann factor for this case is

$$e^{-\beta\left(N\varepsilon - N\mu\right)} = e^{-\beta N(\varepsilon-\mu)} \ . \tag{69}$$

The *average occupation of the energy level* ε can be then calculated as

$$\langle N_\varepsilon \rangle = \frac{\sum_{\substack{\text{all possible} \\ \text{values of N}}} N e^{-\beta N(\varepsilon-\mu)}}{\sum_{\substack{\text{all possible} \\ \text{values of N}}} e^{-\beta N(\varepsilon-\mu)}} \ , \tag{70}$$

where the denominator is the normalization constant for the occupation probability given by the Boltzmann factor.

5.2. The Fermi-Dirac distribution

For fermions occupying a single energy level ε, N can only take two possible values: 0 or 1. Consequently, Eq. (70) becomes

$$\langle N_\varepsilon \rangle_{\text{FD}} = \frac{0 \times e^{-0} + 1 \times e^{-\beta(\varepsilon-\mu)}}{e^{-0} + e^{-\beta(\varepsilon-\mu)}} = \frac{e^{-\beta(\varepsilon-\mu)}}{1 + e^{-\beta(\varepsilon-\mu)}} = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} \ . \tag{71}$$

$\langle N_\varepsilon \rangle_{\text{FD}}$ is the average occupation of a particular energy level ε, and the reasoning can be repeated for *any* energy level, where $\langle N_\varepsilon \rangle_{\text{FD}}$ will take different values for different ε. Thus we have an expression for an average occupation of any energy level ε, which can be also understood in terms of *the probability of occupying the state* ε (if the probability is high, the average occupation is high, and *vice versa*) – i.e., we have just obtained the Fermi-Dirac distribution,

$$f_{\text{FD}} = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} \ . \tag{72}$$

Figure 1 shows the Fermi distribution function for non-interacting nuclear matter at saturation density, calculated at four different temperatures, $T = \{0, 1, 10, 50\}$ MeV, as a function of the particle energy ε. The curve for $T = 0$ shows that all energy states up to the Fermi energy, which at $T = 0$ is equal to the chemical potential, are occupied, while all states above it are empty. The curve at $T = 1$ MeV is virtually indistinguishable from the $T = 0$ curve. At $T = 10$ MeV, the distribution starts to deviate from the zero-temperature behavior, followed by a much clearer deviation at $T = 50$ MeV. We can see that, in general, increasing the temperature populates an increasing share of states above the chemical potential; from Eq. (72), we can see that the latter is equal to the energy ε for which $f(\varepsilon) = 1/2$.

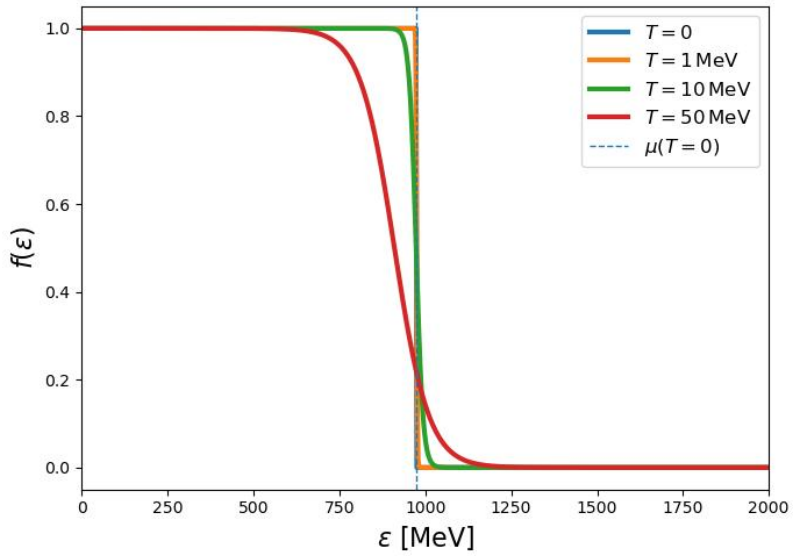

FIG. 1: The Fermi distribution function for non-interacting nuclear matter, for which $\mu(T = 0) \approx 975$ MeV.

5.3. Bonus: the Bose-Einstein distribution

In the case of bosons, N can take any integer value from 0 to infinity,

$$\langle N_\varepsilon \rangle = \frac{\sum_{N=0}^{\infty} N e^{-\beta N(\varepsilon-\mu)}}{\sum_{N=0}^{\infty} e^{-\beta N(\varepsilon-\mu)}} = \frac{\sum_{N=0}^{\infty} N \left[e^{-\beta(\varepsilon-\mu)}\right]^N}{\sum_{N=0}^{\infty} \left[e^{-\beta(\varepsilon-\mu)}\right]^N} \ . \tag{73}$$

For clarity, let us temporarily denote

$$x = e^{-\beta(\varepsilon-\mu)} \ . \tag{74}$$

Then it's easy to see that the denominator in Eq. (73) is an infinite sum over geometric series, equal

$$\sum_{N=0}^{\infty} x^N = \frac{1}{1-x} \ . \tag{75}$$

(Here, the convergence condition is that $|x| < 1$.) In the case of the numerator, let us see that

$$\sum_{N=0}^{\infty} N x^N = \sum_{N=0}^{\infty} x \frac{d}{dx}\left(x^N\right) = x \frac{d}{dx}\left(\sum_{N=0}^{\infty} x^N\right) = x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2} \ , \tag{76}$$

where we have used Eq. (75). Altogether, substituting for x from Eq. (74), Eq. (73) becomes

$$\langle N_\varepsilon \rangle = \frac{e^{-\beta(\varepsilon-\mu)}}{1-e^{-\beta(\varepsilon-\mu)}} = \frac{1}{e^{\beta(\varepsilon-\mu)}-1} \ . \tag{77}$$

By similar arguments as in the case of the Fermi-Dirac distribution, we identify the Bose-Einstein distribution,

$$f_{\mathrm{BE}} = \frac{1}{e^{\beta(\varepsilon-\mu)}-1} \ . \tag{78}$$

5.4. The $T \to 0$ limit of the Fermi-Dirac distribution

Consider the Fermi-Dirac distribution f_{FD} in the case where the temperature is very low, $T \to 0$, which is equivalent to $\beta \to \infty$. The behavior of f_{FD} in this limit strongly depends on the sign of the term $(\varepsilon - \mu)$. If $\varepsilon > \mu$, then the expression in the bracket is positive and the exponent diverges to infinity, so that

$$\lim_{\beta\to\infty} e^{\beta(\varepsilon-\mu)}\Big|_{\varepsilon>\mu} = \infty \ , \tag{79}$$

and consequently

$$\lim_{\beta\to\infty} \frac{1}{e^{\beta(\varepsilon-\mu)}+1}\Big|_{\varepsilon>\mu} = 0 \ . \tag{80}$$

Conversely, if $\varepsilon < \mu$, then the expression in the bracket is negative and the exponent diverges to minus infinity, so that

$$\lim_{\beta\to\infty} e^{\beta(\varepsilon-\mu)}\Big|_{\varepsilon<\mu} = 0 \ , \tag{81}$$

and consequently

$$\lim_{\beta\to\infty} \frac{1}{e^{\beta(\varepsilon-\mu)}+1}\Big|_{\varepsilon<\mu} = 1 \ . \tag{82}$$

Evidently, in the limit $T \to 0$, the Fermi-Dirac distribution is equal exactly 1 up to $\varepsilon = \mu$, after which it's equal 0. This means that all states up to $\varepsilon = \mu$ are occupied, and all states with higher energies are unoccupied. We have already discussed this case at length in Section **??**: the highest occupied energy state is the Fermi energy ε_F, so that we immediately get that the chemical potential is $\mu = \varepsilon_F$. Moreover, for an ideal gas $\varepsilon_F = \sqrt{m^2c^4 + p_F^2 c^2}$, and we

can calculate p_F given the density of the system using Eq. (17). Since all states with $p < p_F$ correspond to $f_{\rm FD} = 1$ while all states with $p > p_F$ correspond to $f_{\rm FD} = 0$, one can rewrite Eqs. (66)-(68) as

$$n\big|_{T=0} = \frac{g}{2\pi^2\hbar^3}\int_0^{p_F} dp\ p^2\ , \tag{83}$$

$$\mathcal{E}\big|_{T=0} = \frac{g}{2\pi^2\hbar^3}\int_0^{p_F} dp\ p^2\ \varepsilon\ , \tag{84}$$

$$P\big|_{T=0} = \frac{gc^2}{6\pi^2\hbar^3}\int_0^{p_F} dp\ \frac{p^4}{E}\ . \tag{85}$$

Naturally, these are just Eqs. (16), (18), and (48).

6. EXERCISES

1. Use the first law of thermodynamics, Eq. (30), to rewrite Eq. (37) using the chemical potential μ. With this, consider how could one have obtained Eq. (41) faster than shown here?

2. Take a brute-force derivative of Eq. (29) and confirm that you get the same result for μ as in these notes, Eq. (41).

3. Show that for fermions, Eq. (65) is equivalent to

$$P = g\int \frac{d^3p}{(2\pi)^3}\ \ k_BT\ \ln\left[1+\exp\Big(-\frac{\varepsilon-\mu}{k_BT}\Big)\right]\ . \tag{86}$$

   Pressure can be also cast in a similar form for bosons. What is it?

4. What is the high-temperature limit of the Fermi-Dirac distribution? What about the Bose distribution?

5. The formulas derived here describe one species of particles. How will they change if one considers multiple non-interacting species? What formula should you use for non-interacting symmetric nuclear matter?

6. Consider the expression for the number density n, Eq. (66), in the high-temperature limit. Given n and T, how can you calculate μ? Assuming we are dealing with non-interacting symmetric nuclear matter, plot μ as a function of n (in the range from 0 to 6 times saturation density, $n_0 = 0.160$ fm^{-3}) for $T = \{1,\ 10,\ 50,\ 100,\ 200\}$ MeV. (Note: for the first few values of T, the results will be wrong as for these T one should use the low-temperature distribution.)

7. Given n, for Fermi and Bose distributions at a finite T the chemical potential can only be obtained by solving a root equation. Here, one uses the fact that

$$n = g\int \frac{d^3p}{(2\pi)^3}\ \ f(p,T,\mu) \tag{87}$$

   is only true if a correct value of μ is used. The root equation is obtained by putting all terms in the above equation on one side. Write a function calculating μ at given n and T, then use that to plot μ as a function of n for $T = \{1,\ 10,\ 50,\ 100,\ 200\}$ MeV. How do your results compare with what you got in the previous exercise?

Appendix A: Motivating Eq. (47) using the kinetic theory

In most basic terms, pressure is force per unit area. Since force is given by the change in momentum per unit time, $F = ma = m\frac{\Delta v}{\Delta t} = \frac{\Delta p}{\Delta t}$, we have

$$P = \frac{\Delta p}{\Delta A \Delta t} \ . \tag{A1}$$

In the kinetic theory, the pressure can be defined as the average momentum flux, i.e., as the amount of momentum that particles impart on the walls containing the system. These walls may be either real or hypothetical – for example, to establish the pressure in the middle of some continuous medium, one imagines a hypothetical wall in the middle and considers how much momentum would be imparted on that wall by the system.

To make the connection to the basic definition from Eq. (A1), let us consider this hypothetical wall perpendicular to the x-axis. How many particles will hit the area of the wall spanned by ΔA in time Δt? Let us first assume that all particles in the vicinity of ΔA have the same velocity v_x. Then all particles that are maximally $v_x \Delta t$ away from ΔA will hit it in time Δt. This allows us to define a volume containing all particles that will hit ΔA in time Δt:

$$\Delta V = (v_x \Delta t)\Delta A \ . \tag{A2}$$

How many particles are in that volume? If the local density is n, then

$$N = n\Delta V = n v_x \Delta t \Delta A \ . \tag{A3}$$

Then the number of particles hitting the wall per unit area per unit time is

$$\frac{N}{\Delta A \Delta t} = n v_x \ , \tag{A4}$$

where nv_x is a flux of particles. Of course, we have simplified the situation by assuming that all particles in ΔV move with the same velocity. This is not the case. Instead, the particles momenta (and therefore velocities) are distributed according to some distribution f. Therefore, the number density of particles with momentum p_x, $v_x = p_x c/E$, is given by $d^3p \ f(p = p_x)$. By integrating over all particles having all possible velocities, Eq. (A3) becomes

$$N = g \int_{p_x > 0} \frac{d^3p}{(2\pi\hbar)^3} \ v_x \Delta t \Delta A \ f \ . \tag{A5}$$

Now, upon hitting the wall, assuming the particles bounce off elastically, each of these particles transfers an amount of momentum $\Delta p = 2p_x$ to the wall. Thus, the total amount of transferred momentum is

$$\Delta p_{\text{wall}} = \Delta A \Delta t \ g \int_{p_x > 0} \frac{d^3p}{(2\pi\hbar)^3} \ (2p_x) v_x \ f \ . \tag{A6}$$

Since in equilibrium we expect f to be isotropic, we can take into the account the factor of 2 in the integrand by removing the condition $p_x > 0$ from the integration region, thus arriving at

$$\Delta p_{\text{wall}} = \Delta A \Delta t \ g \int \frac{d^3p}{(2\pi\hbar)^3} \ p_x v_x \ f = \Delta A \Delta t \ g \int \frac{d^3p}{(2\pi\hbar)^3} \ \frac{p_x^2 c}{E} \ f = \Delta A \Delta t \ \left\langle \frac{p_x^2 c}{E} \right\rangle \ . \tag{A7}$$

In equilibrium, we can also expect that the average of $p_x^2 c/E$ equals the average of $p_y^2 c/E$ and $p_z^2 c/E$, thus we can write

$$\Delta p_{\text{wall}} = \Delta A \Delta t \ \frac{g}{3} \int \frac{d^3p}{(2\pi\hbar)^3} \ \frac{(p_x^2 + p_y^2 + p_z^2)c}{E} \ f = \Delta A \Delta t \ g \int \frac{d^3p}{(2\pi\hbar)^3} \ \frac{p^2 c}{3E} \ f = \Delta A \Delta t \ g \int \frac{d^3p}{(2\pi\hbar)^3} \ \frac{pv}{3} \ f \ . \tag{A8}$$

Using Eq. (A1), this yields Eq. (47),

$$P = g \int \frac{d^3p}{(2\pi)^3} \ \frac{pv}{3} \ f \ , \tag{A9}$$

where $dn = g\frac{d^3p}{(2\pi\hbar)^3} \ f$. Given that

$$g \int \frac{d^3p}{(2\pi)^3} \ \ v \ f \tag{A10}$$

can be identified is particle flux, it follows that $g \int \frac{d^3p}{(2\pi)^3} \ pv \ f$ is a *momentum flux*. The spatial diagonal components of an energy-momentum tensor are momentum fluxes in the x, y, and z directions, which is why Eq. (47) can be obtained using an energy momentum tensor.

Appendix B: Exercise solutions

Exercise 1

Use the first law of thermodynamics, Eq. (30), to rewrite Eq. (37) using the chemical potential μ. With this, consider how could one have obtained Eq. (41) faster than shown here?

The first law of thermodynamics is, Eq. (30),

$$dE = TdS - PdV + \mu dN \ , \tag{B1}$$

while the pressure at $T = 0$ is given by

$$P\big|_{T=0} = n\frac{d\mathcal{E}}{dn} - \mathcal{E} \ . \tag{B2}$$

Assuming $V = \text{const}$, the first law becomes

$$d\left(\frac{E}{V}\right) = T\ d\left(\frac{S}{V}\right) + \mu\ d\left(\frac{N}{V}\right) \ , \tag{B3}$$

which can be rewritten as

$$d\mathcal{E} = T\ ds + \mu\ dn \ . \tag{B4}$$

Then it follows that

$$\left.\frac{d\mathcal{E}}{dn}\right|_s = \mu \ . \tag{B5}$$

Since $s = \text{const}$ is automatically satisfied at $T = 0$, we can rewrite Eq. (37) as

$$P = n\mu - \mathcal{E} \ . \tag{B6}$$

Since at $T = 0$, $\mu = \varepsilon_F$, this immediately yields Eq. (41).

If you are careful, you should be a little worried about the argument above. We have derived Eq. (37) by assuming $N = \text{const}$, but to obtain Eq. (B5) we assumed $V = \text{const}$. Luckily, Eq. (37) can also be derived without invoking that assumption. Indeed, the Euler relation states that

$$E = TS - PV + \mu N \ . \tag{B7}$$

(This relation is obtained from the first law of thermodynamics under the assumption that S, V, and N change linearly as the system size is increased.) We can rewrite the above as

$$PV = TS + \mu N - E \ , \tag{B8}$$

which divided by V and at $T = 0$ immediately yields Eq. (37),

$$P = \mu n - \mathcal{E} \ . \tag{B9}$$

Exercise 2

Take a brute-force derivative of Eq. (29) and confirm that you get the same result for μ as in these notes, Eq. (41).

The energy density at $T = 0$, given by Eq. (29), repeated here for convenience,

$$\mathcal{E}\big|_{T=0} = \frac{g}{16\pi^2\hbar^3}\left[\frac{2\varepsilon_F^3 p_F}{c^2} - \varepsilon_F p_F m^2c^2 - m^4c^5 \ln\left(\frac{\varepsilon_F + p_F c}{mc^2}\right)\right] , \tag{B10}$$

depends on only only one thermodynamic variable: density. We have $\varepsilon_F = \varepsilon_F(p_F)$ and $p_F = p_F(n)$, so that the differential of p_F with respect to n is

$$dp_F \equiv \frac{dp_F}{dn}\, dn = \frac{1}{3}\left(\frac{6\pi^2\hbar^3 n}{g}\right)^{-2/3} \frac{6\pi^2\hbar^3}{g}\, dn = \frac{1}{3}\left(\frac{6\pi^2\hbar^3 n}{g}\right)^{-2/3} \frac{1}{n}\left(\frac{6\pi^2\hbar^3 n}{g}\right)\, dn = \frac{p_F}{3n}\, dn\ . \tag{B11}$$

Consequently,

$$\frac{d\mathcal{E}\big|_{T=0}}{dn} = \frac{p_F}{3n}\frac{d\mathcal{E}\big|_{T=0}}{dp_F}\ . \tag{B12}$$

We can calculate

$$\frac{d\mathcal{E}\big|_{T=0}}{dp_F} = \frac{g}{16\pi^2\hbar^3}\left[\frac{6\varepsilon_F^2 \frac{d\varepsilon_F}{dp_F} p_F}{c^2} + \frac{2\varepsilon_F^3}{c^2} - \frac{d\varepsilon_F}{dp_F} p_F m^2c^2 - \varepsilon_F m^2c^2 - m^4c^5 \frac{\frac{d\varepsilon_F}{dp_F} + c}{\varepsilon_F + p_F c}\right]\ . \tag{B13}$$

Since

$$\frac{d\varepsilon_F}{dp_F} = \frac{p_F c^2}{\varepsilon_F}\ , \tag{B14}$$

we obtain

$$\begin{aligned}
\frac{d\mathcal{E}\big|_{T=0}}{dp_F} &= \frac{g}{16\pi^2\hbar^3}\left[\frac{6\varepsilon_F p_F^2 c^2}{c^2} + \frac{2\varepsilon_F^3}{c^2} - \frac{p_F^2 m^2c^4}{\varepsilon_F} - \varepsilon_F m^2c^2 - m^4c^5 \frac{\frac{c}{\varepsilon_F}(p_F c + \varepsilon_F)}{\varepsilon_F + p_F c}\right] && \text{(B15)}\\
&= \frac{g}{16\pi^2\hbar^3}\left[6\varepsilon_F p_F^2 + \frac{2\varepsilon_F(p_F^2c^2 + m^2c^4)}{c^2} - \frac{p_F^2 m^2c^4}{\varepsilon_F} - \varepsilon_F m^2c^2 - \frac{m^4c^6}{\varepsilon_F}\right] && \text{(B16)}\\
&= \frac{g}{16\pi^2\hbar^3}\left[6\varepsilon_F p_F^2 + 2\varepsilon_F p_F^2 + 2\varepsilon_F m^2c^2 - \varepsilon_F m^2c^2 - m^2c^2\left(\frac{p_F^2c^2}{\varepsilon_F} + \frac{m^2c^4}{\varepsilon_F}\right)\right] && \text{(B17)}\\
&= \frac{g}{16\pi^2\hbar^3}\left[8\varepsilon_F p_F^2 + \varepsilon_F m^2c^2 - m^2c^2\left(\frac{p_F^2c^2}{\varepsilon_F} + \frac{m^2c^4}{\varepsilon_F}\right)\right] && \text{(B18)}\\
&= \frac{g}{16\pi^2\hbar^3}\left[8\varepsilon_F p_F^2\right] = \frac{g}{2\pi^2\hbar^3}\varepsilon_F p_F^2\ , && \text{(B19)}
\end{aligned}$$

and from that, using Eq. (B12),

$$\frac{d\mathcal{E}\big|_{T=0}}{dn} = \frac{g}{6\pi^2\hbar^3 n}\varepsilon_F p_F^3 = \varepsilon_F\ , \tag{B20}$$

which is the same as Eq. (41).

Exercise 3

Show that for fermions, Eq. (65) is equivalent to

$$P = g \int \frac{d^3p}{(2\pi)^3} \; k_B T \; \ln\left[1 + \exp\Big(-\frac{\varepsilon - \mu}{k_B T}\Big)\right] . \tag{B21}$$

Pressure can be also cast in a similar form for bosons. What is it?

The ideal Fermi gas pressure, Eq. (65),

$$P = \frac{g}{3} \int \frac{d^3p}{(2\pi\hbar)^3} \; pv \; f = \frac{g}{3} \int \frac{d^3p}{(2\pi\hbar)^3} \; \frac{p^2c^2}{\varepsilon} \; \frac{1}{e^{\beta\left(\varepsilon-\mu\right)} + 1} , \tag{B22}$$

can be rewritten as

$$P = -\frac{g}{3} \int \frac{d^3p}{(2\pi\hbar)^3} \; \frac{p^2c^2}{\varepsilon} \; \Big(\beta\frac{pc^2}{\varepsilon}\Big)^{-1} \frac{d}{dp}\left(\ln\left[1 + e^{-\beta\left(\varepsilon-\mu\right)}\right]\right) \tag{B23}$$

$$= -g \int \frac{dp}{(2\pi\hbar)^3} \; \beta^{-1} \; \frac{p^3}{3} \; \frac{d}{dp}\left(\ln\left[1 + e^{-\beta\left(\varepsilon-\mu\right)}\right]\right) \tag{B24}$$

$$= g \int \frac{dp}{(2\pi\hbar)^3} \; \left[\frac{d}{dp}\left(\beta^{-1} \; \frac{p^3}{3}\right)\right] \; \left(\ln\left[1 + e^{-\beta\left(\varepsilon-\mu\right)}\right]\right) \tag{B25}$$

$$= g \int \frac{dp}{(2\pi\hbar)^3} \; \beta^{-1} \; p^2 \; \left(\ln\left[1 + e^{-\beta\left(\varepsilon-\mu\right)}\right]\right) \tag{B26}$$

$$= g \int \frac{d^2p}{(2\pi\hbar)^3} \; k_B T \; \ln\left[1 + e^{-\beta\left(\varepsilon-\mu\right)}\right] , \tag{B27}$$

where in the third equality we used integration by parts and used the fact that the integrand is zero both at $p = 0$ and at $p = \infty$ to get rid of the definite integral term.

For the Bose gas, the pressure is given by

$$P_{\text{Bose}} = \frac{g}{3} \int \frac{d^3p}{(2\pi\hbar)^3} \; pv \; f_{\text{Bose}} = \frac{g}{3} \int \frac{d^3p}{(2\pi\hbar)^3} \; \frac{p^2c^2}{\varepsilon} \; \frac{1}{e^{\beta\left(\varepsilon-\mu\right)} - 1} . \tag{B28}$$

Repeating the steps as above,

$$P_{\text{Bose}} = \frac{g}{3} \int \frac{d^3p}{(2\pi\hbar)^3} \; \frac{p^2c^2}{\varepsilon} \; \Big(\beta\frac{pc^2}{\varepsilon}\Big)^{-1} \frac{d}{dp}\left(\ln\left[1 - e^{-\beta\left(\varepsilon-\mu\right)}\right]\right) \tag{B29}$$

$$= g \int \frac{dp}{(2\pi\hbar)^3} \; \beta^{-1} \; \frac{p^3}{3} \; \frac{d}{dp}\left(\ln\left[1 - e^{-\beta\left(\varepsilon-\mu\right)}\right]\right) \tag{B30}$$

$$= -g \int \frac{dp}{(2\pi\hbar)^3} \; \left[\frac{d}{dp}\left(\beta^{-1} \; \frac{p^3}{3}\right)\right] \; \left(\ln\left[1 - e^{-\beta\left(\varepsilon-\mu\right)}\right]\right) \tag{B31}$$

$$= -g \int \frac{dp}{(2\pi\hbar)^3} \; \beta^{-1} \; p^2 \; \left(\ln\left[1 - e^{-\beta\left(\varepsilon-\mu\right)}\right]\right) \tag{B32}$$

$$= -g \int \frac{d^2p}{(2\pi\hbar)^3} \; k_B T \; \ln\left[1 - e^{-\beta\left(\varepsilon-\mu\right)}\right] . \tag{B33}$$

Exercise 4

What is the high-temperature limit of the Fermi-Dirac distribution? What about the Bose distribution?

The Fermi and Bose distribution, both of which give a probability for a state of energy ε (or equivalently momentum p) to be occupied, can be considered at the same time by writing

$$f_{\pm} = \frac{1}{e^{\frac{\varepsilon-\mu}{k_B T}} \pm 1} \ , \tag{B34}$$

where the upper sign (plus) is for the Fermi distribution and the lower sign (minus) is for the Bose distribution. In the high-temperature limit, there are generally more states accessible than at lower temperature. Therefore, particles are spread out over many states and the probability of occupying any single state is small,

$$f_{\pm} = \frac{1}{e^{\frac{\varepsilon-\mu}{k_B T}} \pm 1} \ll 1 \ . \tag{B35}$$

This is equivalent to

$$e^{\frac{\varepsilon-\mu}{k_B T}} \gg 1 \ . \tag{B36}$$

leading to

$$\lim_{T \gg 1} f_{\pm} \approx \frac{1}{e^{\frac{\varepsilon-\mu}{k_B T}}} = e^{-\beta(\varepsilon-\mu)} \ , \tag{B37}$$

i.e., the Boltzmann distribution.

Exercise 5

The formulas derived here describe one species of particles. How will they change if one considers multiple non-interacting species? What formula should you use for non-interacting symmetric nuclear matter?

Let us consider the number density of some baryonic species indexed by i. We have

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} \frac{1}{\exp\left[\beta(\varepsilon_i - \mu_i)\right] + 1} , \tag{B38}$$

where (for an ideal gas)

$$\varepsilon_i = \sqrt{m_i^2 + p^2} . \tag{B39}$$

If we have multiple species, the total density is the sum of contributing densities,

$$n_{\text{total}} = \sum_i n_i = \sum_i g_i \int \frac{d^3p}{(2\pi)^3} \frac{1}{\exp\left[\beta(\varepsilon_i - \mu_i)\right] + 1} . \tag{B40}$$

Note that in principle, each species' chemical potential can be different, and their values depend on the partial densities n_i.

In a non-interacting symmetric nuclear matter, we have equal amounts of protons and neutrons, so that the proton and neutron density are equal, $n_N = n_P$. Since to a good approximation the masses of protons and neutrons are the same, $m_N \approx m_P$, then we'll also have $\mu_N \approx \mu_P$. Denoting the nucleon mass simply by m and the chemical potential by μ, we then have

$$n_N = n_P = g \int \frac{d^3p}{(2\pi)^3} \frac{1}{\exp\left[\beta(\varepsilon - \mu)\right] + 1} , \tag{B41}$$

where $\varepsilon = \sqrt{m^2 + p^2}$. Then the total density, which in this case is equivalent to the total baryon density n_B, is

$$n_B = n_N + n_P = 2 \times g \int \frac{d^3p}{(2\pi)^3} \frac{1}{\exp\left[\beta(\varepsilon - \mu)\right] + 1} . \tag{B42}$$

As a result, since the expressions for the neutron and proton densities are degenerate in symmetric nuclear matter, one often redefines the degeneracy factor g to include degeneracy due to isospin,

$$\tilde{g} = g g_{\text{isospin}} . \tag{B43}$$

Here, g is the usual degeneracy factor due to the particles having a spin s, g = 2s+1, which for nucleons equals 2, while g_{isospin} takes care of the factor of 2 from Eq. (B42), so that $\tilde{g} = 4$ and we can write

$$n_B = n_N + n_P = \tilde{g} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\exp\left[\beta(\varepsilon - \mu)\right] + 1} . \tag{B44}$$

Usually, one omits the tilde mark over $\tilde{g}$ and writes simply g, with the understanding that for symmetric nuclear matter $g = 4$.

Exercise 6

Consider the expression for the number density n, Eq. (66), in the high-temperature limit. Given n and T, how can you calculate μ? Assuming we are dealing with non-interacting symmetric nuclear matter, plot μ as a function of n (in the range from 0 to 6 times saturation density, $n_0 = 0.160$ fm^{-3}) for $T = \{1,\ 10,\ 50,\ 100,\ 200\}$ MeV. (Note: for the first few values of T, the results will be wrong as for these T one should use the low-temperature distribution.)

In the high-temperature limit, the distribution f becomes the Boltzmann distribution and

$$n = g \int \frac{d^3p}{(2\pi)^3}\ e^{-\beta(\varepsilon-\mu)}\ . \tag{B45}$$

Since μ is not a function of the momentum p, we can take the exponent in front of the integral,

$$n = e^{\beta\mu}\ g \int \frac{d^3p}{(2\pi)^3}\ e^{-\beta\varepsilon}\ . \tag{B46}$$

The integral in the equation above can be performed analytically, yielding something known as the modified Bessel function of the second kind K. Alternatively, one can also perform the integral numerically. Either way, denoting

$$g \int \frac{d^3p}{(2\pi)^3}\ e^{-\beta\varepsilon} = \mathcal{I}\ , \tag{B47}$$

we see that we can solve for μ,

$$\mu = T \ln\left(\frac{n}{\mathcal{I}}\right)\ . \tag{B48}$$

Exercise 7

Given n, for Fermi and Bose distributions at a finite T the chemical potential can only be obtained by solving a root equation. Here, one uses the fact that

$$n = g \int \frac{d^3p}{(2\pi)^3} \quad f(p, T, \mu) \tag{B49}$$

is only true if a correct value of μ is used. The root equation is obtained by putting all terms in the above equation on one side. Write a function calculating μ at given n and T, then use that to plot μ as a function of n for $T = \{1,\ 10,\ 50,\ 100,\ 200\}$ MeV. How do your results compare with what you got in the previous exercise?

N/A.